A VERSION OF GORDON'S THEOREM
FOR MULTI-DIMENSIONAL SCHRÖDINGER OPERATORS

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ABSTRACT. We consider discrete Schrödinger operators $H = \Delta + V$ in $\ell^2(\mathbb{Z}^d)$ with $d \geq 1$, and study the eigenvalue problem for these operators. It is shown that the point spectrum is empty if the potential V is sufficiently well approximated by periodic potentials. This criterion is applied to quasiperiodic V and to so-called Fibonacci-type superlattices.

1. INTRODUCTION

In this article we are interested in the eigenvalue problem for discrete Schrödinger operators

$$(1) \qquad (H\phi)(n) = \sum_{|n-m|_1=1} \phi(m) + V(n)\phi(n)$$

in the Hilbert space $\ell^2(\mathbb{Z}^d)$, where $V : \mathbb{Z}^d \to \mathbb{R}$ and for $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, we set $|k|_1 = |k_1| + \cdots + |k_d|$. That is, we shall study the difference equation

$$(2) \qquad \sum_{|n-m|_1=1} \psi(m) + V(n)\psi(n) = E\psi(n)$$

and the question of whether or not this equation admits nontrivial solutions in $\ell^2(\mathbb{Z}^d)$. In the one-dimensional case, $d = 1$, there is a simple albeit brilliant criterion for absence of square-summable solutions which is due to Gordon [12]. Essentially, it states that if the potential is close to being periodic in a suitable sense, then the operator H has empty point spectrum. Intuitively, this is quite reasonable, since (2) has no nontrivial ℓ^2-solutions if V is periodic, and if the potential is close to being periodic, the solutions to (2) should retain this behavior. Of course it is crucial in which sense the potential needs to be close to a periodic one, and this will be a major issue in this paper, in particular when we discuss applications in later sections. Let us mention that criteria of this flavor and their applications in one dimension have been reviewed in [8]. It is safe to say that Gordon's idea has led to many, if not most, key continuity results for Schrödinger operators in one dimension, and hence can be regarded as a central tool in the spectral theory of such operators. However, the proof of the Gordon criterion uses one-dimensional

techniques, and it is not quite clear whether there is any hope of finding an analog in higher dimensions.

However, another paper by Gordon [13] has dealt with this issue, that is, studying the multi-dimensional case, $d > 1$. It does establish a criterion in this spirit for absence of eigenvalues, but there is a major pitfall: While the one-dimensional criterion of [12] requires *local* approximability by periodic potentials, the criterion proved in [13] requires uniform *global* approximability. This has striking consequences for the applicability of the criterion. The one-dimensional criterion is most noted for its applications to quasi-periodic potentials, for example, classes of potentials containing the almost Mathieu potential or the Fibonacci potential. On the other hand, the criterion of [13] cannot be applied to the higher-dimensional analogs of such potentials. It is easy to see that they do not satisfy the assumption of uniform global approximability by periodic potentials.

This leads us to our main motivation. We want to find a criterion in Gordon's spirit that establishes empty point spectrum for multi-dimensional Schrödinger operators which can be applied to some quasi-periodic potentials. After some preparatory work in Section 2, we will present such a criterion in Section 3. This criterion requires periodicity of V in $d-1$ directions and *local* approximability by periodic potentials in the remaining direction. Applications of this result to quasi-periodic V will be presented in Sections 4 and 5. In Section 6 we discuss the method used in Section 3.

2. Gordon's theorem for quasi-one-dimensional operators

As a warm-up, and also to provide a tool we will use in the next section, we consider quasi-one-dimensional operators and prove a basic Gordon theorem for them. This result is similar to the treatment of operators on the strip presented in [9]. In fact, we will reduce our problem at hand to the case of the strip.

Rather than considering operators acting in $\ell^2(\mathbb{Z}^d)$ we will replace $\mathbb{Z}^d$ by a *cylinder* in $\mathbb{Z}^d$. Thus, picking numbers $p_1, \ldots, p_{d-1}$, we consider the cylinder $C = \{1, \ldots, p_1\} \times \cdots \times \{1, \ldots, p_{d-1}\} \times \mathbb{Z}$ and the Hilbert space $\ell^2(C)$. In principle, we want to consider Schrödinger operators in this Hilbert space, using suitable boundary conditions. But this can be studied within the standard strip framework. Namely, defining $l = \prod_{k=1}^{d-1} p_k$, this problem is equivalent to a problem on a strip in $\mathbb{Z}^2$ of width l.

Once in this framework, we continue as follows: The discrete Schrödinger operator H_l on the strip of width l acts, in the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^l)$, on vector sequences with l complex components $\varphi(n) = (\varphi_1(n), \ldots, \varphi_l(n))^T$ by the formula

$$(3) \qquad (H_l \varphi)(n) = \varphi(n+1) + \varphi(n-1) + V(n)\varphi(n).$$

Here, V is a mapping $V : \mathbb{Z} \to \{M \in \mathbb{R}^{l,l} : M \text{ symmetric}\}$. Note that the transformation from cylinder in $\mathbb{Z}^d$ to a strip in $\mathbb{Z}^2$ yields an operator of this form. Let us consider solutions ψ of the difference equation

$$(4) \qquad \psi(n+1) + \psi(n-1) + V(n)\psi(n) = E\psi(n).$$

Define, for $n \in \mathbb{Z}$, the $(2l)$-dimensional vector $\Psi(n)$ by

$$(5) \qquad \Psi(n) = \begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix}.$$

Theorem 1. *Suppose there are $n_k \in \mathbb{N}$ such that $n_k \to \infty$ and for every k we have, for every $-2l + 1 \le m \le 2l - 1$ and every $1 \le j \le n_k$,*

$$(6) \qquad V(mn_k + j) = V(j).$$

Then the operator H_l in (3) has empty point spectrum. More specifically, we have, for every energy E, every solution ψ to (4), and every $k \in \mathbb{N}$, the uniform estimate

$$(7) \qquad \max\{\, \|\Psi((-2l+1)n_k)\|, \|\Psi((-2l+2)n_k)\|, \ldots, \|\Psi(-n_k)\|,$$

$$\|\Psi(n_k)\|, \ldots, \|\Psi((2l-1)n_k)\|, \|\Psi(2ln_k)\| \,\} \ge \frac{\|\Psi(0)\|}{2l}.$$

Proof. This theorem can be found in [9]. For the reader's convenience, we sketch the argument briefly. Define, for each real number E and each $n \in \mathbb{Z}$, the one-step transfer matrix $T_E(n) \in SL(2l, \mathbb{R})$ by

$$(8) \qquad T_E(n) = \begin{pmatrix} EI_l - V(n) & -I_l \\ I_l & 0 \end{pmatrix},$$

where I_l is the identity matrix of order l. The transfer matrix $M_E(n)$ is then given by

$$(9) \qquad M_E(n) = \begin{cases} T_E(n) \times \cdots \times T_E(1) & \text{if} \quad n \ge 1, \\ I_{2l} & \text{if} \quad n = 0, \\ T_E(n+1)^{-1} \times \cdots \times T_E(0)^{-1} & \text{if} \quad n \le -1. \end{cases}$$

Let ψ be a nontrivial solution to (4). Then we have

$$(10) \qquad \Psi(n) = M_E(n)\Psi(0)$$

for every $n \in \mathbb{Z}$. Observe that condition (6) implies, for every $k \in \mathbb{N}$ and $-2l + 1 \le m \le 2l - 1$,

$$(11) \qquad M_E(mn_k) = M_E(n_k)^m.$$

Let $p(z) = \sum_{j=0}^{2l} a_j z^j$ be the characteristic polynomial of $M_E(n_k)$, where the dependence of the a_j on E, k is left implicit. We always have $a_{2l} = a_0 = 1$. Moreover, by the Cayley-Hamilton theorem, we have

$$(12) \qquad M_E(n_k)^{2l} + a_{2l-1}M_E(n_k)^{2l-1} + \cdots + a_1 M_E(n_k) + I_{2l} = 0.$$

Now choose the smallest J such that $|a_J|$ is maximal among the $|a_j|$ (hence $0 \le J \le 2l - 1$) and apply the matrix on the left hand side of (12) to the vector $\Psi(-Jn_k)$. Then, using (10), (11), and (12), we get

$$\Psi(-Jn_k) + a_1\Psi((-J+1)n_k) + \cdots + a_J\Psi(0) + \cdots + \Psi((2l-J)n_k) = 0.$$

This gives the estimate (7). $\qquad\qquad \square$

3. A version of Gordon's theorem for multi-dimensional operators

In this section we prove the main result of this article, namely, a partially local Gordon-type criterion for multi-dimensional Schrödinger operators. It is a more elaborate version of the main result of [13]. As mentioned in the introduction, we are aiming at a criterion which can be applied to quasi-periodic potentials, and we will present applications of Theorem 2 below to such potentials in subsequent sections.

Theorem 2. *Let H be an operator of the form* (1) *with a bounded potential V. Suppose that*

(i) *there are numbers $p_1, \ldots, p_{d-1} \in \mathbb{N}$ such that*

$$V(n - p_k e_k) = V(n)$$

for $1 \leq k \leq d - 1$ and every $n \in \mathbb{Z}^d$, where $e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$ with the 1 in the k-th entry, and

(ii) *there are numbers $p_d^{(m)} \in \mathbb{N}$ satisfying $p_d^{(m)} \to \infty$ as $m \to \infty$ and for every m there exists a $(p_1, \ldots, p_{d-1}, p_d^{(m)})$-periodic $V^{(m)} : \mathbb{Z}^d \to \mathbb{R}$ such that with $l = \prod_{k=1}^{d-1} p_k$, we have*

$$\sup_{\substack{n=(n_1,\ldots,n_d)\in\mathbb{Z}^d: \\ (-2l+1)p_d^{(m)} \leq n_d \leq (2l)p_d^{(m)}}} |V(n) - V^{(m)}(n)| \leq Cm^{-p_d^{(m)}}$$

for some m-independent constant C. Then for every E, the difference equation (2) *has no nontrivial solution in $\ell^2(\mathbb{Z}^d)$, and hence the operator H has empty point spectrum.*

Proof. Consider, for some arbitrary E, a solution ψ of the difference equation (2) and assume that it belongs to $\ell^2(\mathbb{Z}^d)$. Our goal is to show $\psi = 0$. With the numbers $p_1, \ldots, p_{d-1}$ from assumption (i) define the cylinder $C = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{d-1}} \times \mathbb{Z}$. With the one-dimensional torus $\mathbb{T} \cong [0, 2\pi)$, let $\theta = (\theta_1, \ldots, \theta_{d-1}) \in \mathbb{T}^{d-1}$, $L \in \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{d-1}}$, and $n_d \in \mathbb{Z}$. Define

$$\zeta(L, n_d, \theta) = \sum_{(n_1,\ldots,n_{d-1})\in L} \psi(n_1, \ldots, n_{d-1}, n_d) e^{i(n_1\theta_1 + \cdots n_{d-1}\theta_{d-1})}.$$

For fixed (L, n_d), this series converges for almost every $\theta \in \mathbb{T}^{d-1}$ since ψ is assumed to be square-summable. Thus, for almost every $\theta \in \mathbb{T}^{d-1}$, this series converges for every pair $(L, n_d) \in C$. In the following, let θ belong to this set of full measure. Using assumption (i), V naturally induces a function $\tilde{V}$ on C by $\tilde{V}(L, n_d) = V(n_1, \ldots, n_{d-1}, n_d)$ for any $(n_1, \ldots, n_{d-1}) \in L$. Writing $\bar{n} = (n_1, \ldots, n_{d-1})$, we get

$$(E - \tilde{V}(L, n_d))\zeta(L, n_d, \theta) = (E - \tilde{V}(L, n_d)) \sum_{\bar{n}\in L} \psi(\bar{n}, n_d) e^{i(\bar{n}\cdot\theta)}$$

$$= \sum_{\bar{n}\in L} \sum_{|(\bar{n},n_d)-m|_1=1} \psi(m) e^{i(\bar{n}\cdot\theta)}$$

$$= \zeta(L, n_d + 1, \theta) + \zeta(L, n_d - 1, \theta) + \cdots$$

$$\cdots + \sum_{j=1}^{d-1} \left(\zeta(L + \delta_j, n_d, \theta) e^{-i\theta_j} + \zeta(L - \delta_j, n_d, \theta) e^{i\theta_j} \right),$$

where $\delta_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{d-1}}$ with the 1 in the j-th entry. This can be rewritten as

$$(13) \qquad \zeta(L, n_d + 1, \theta) + \zeta(L, n_d - 1, \theta) + \sum_{j=1}^{d-1}(\zeta(L + \delta_j, n_d, \theta)e^{-i\theta_j} + \cdots$$

$$\cdots + \zeta(L - \delta_j, n_d, \theta)e^{i\theta_j}) + \tilde{V}(L, n_d)\zeta(L, n_d, \theta) = E\zeta(L, n_d, \theta),$$

which is the difference equation of an operator on the cylinder C. Thus we can study ζ using the method described in the previous section. However, $\tilde{V}$ does not

quite have the locally repetitive structure which is required by Theorem 1. We therefore use the approximants $\tilde{V}^{(m)}$ that are suggested by assumption (ii), that is, the potentials induced by the periodic $V^{(m)}$, and study the difference equation associated with them. Consider the solution $\xi^{(m)}$ of

$$(14) \quad \xi^{(m)}(L, n_d + 1, \theta) + \xi^{(m)}(L, n_d - 1, \theta) + \sum_{j=1}^{d-1} (\xi^{(m)}(L + \delta_j, n_d, \theta)e^{-i\theta_j} + \cdots$$

$$\cdots + \xi^{(m)}(L - \delta_j, n_d, \theta)e^{i\theta_j}) + \tilde{V}^{(m)}(L, n_d)\xi^{(m)}(L, n_d, \theta) = E\xi^{(m)}(L, n_d, \theta)$$

which satisfies

$$(15) \qquad\qquad \xi^{(m)}(L, 0, \theta) = \zeta(L, 0, \theta).$$

Let $\bar{\xi}^{(m)}(n_d, \theta)$ be the $2l$-dimensional vector whose entries are $\xi^{(m)}(L, n_d, \theta)$, $\xi^{(m)}(L, n_d + 1, \theta)$, $L \in \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{d-1}}$, where we choose an arbitrary, but fixed, ordering of the elements of $(\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{d-1}}) \times \{0, 1\}$. It follows from Theorem 1 and the discussion preceding it that

(16)

$$\max \{ \|\bar{\xi}^{(m)}((-2l+1)p_d^{(m)}, \theta)\|, \|\bar{\xi}^{(m)}((-2l+2)p_d^{(m)}, \theta)\|, \ldots, \|\bar{\xi}^{(m)}(-p_d^{(m)}, \theta)\|,$$

$$\|\bar{\xi}^{(m)}(p_d^{(m)}, \theta)\|, \ldots, \|\bar{\xi}^{(m)}((2l-1)p_d^{(m)}, \theta)\|, \|\bar{\xi}^{(m)}(2lp_d^{(m)}, \theta)\| \} \geq \frac{\|\bar{\xi}^{(m)}(0, \theta)\|}{2l}.$$

Similarly, let $\bar{\zeta}(n_d, \theta)$ be the $2l$-dimensional vector whose entries are $\zeta(L, n_d, \theta)$, $\zeta(L, n_d + 1, \theta)$, $L \in \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{d-1}}$, where we choose the same ordering of the elements of $(\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{d-1}}) \times \{0, 1\}$ as above.

Let $I^{(m)} = \{(-2l+1)p_d^{(m)}, \ldots, (2l)p_d^{(m)}\}$. Using the transfer matrices associated with (13) and (14), as introduced in the proof of Theorem 1, and a telescoping sum argument as in [6] and [12], it follows that

$$(17) \qquad \sup_{n_d \in I^{(m)}} |\bar{\zeta}(n_d, \theta) - \bar{\xi}^{(m)}(n_d, \theta)| \quad \leq \quad \sup_{n_d \in I^{(m)}} |n_d| C_l^{|n_d|} m^{-p_d^{(m)}}$$

$$= \quad (2l)p_d^{(m)} C_l^{(2l)p_d^{(m)}} m^{-p_d^{(m)}}$$

$$\to \quad 0 \quad \text{as } m \to \infty.$$

Thus, by (15), (16), (17), and square-summability of $\bar{\zeta}(\cdot, \theta)$, we obtain $\|\bar{\zeta}(0, \theta)\| = \|\bar{\zeta}(1, \theta)\| = 0$. Integrating over θ, we see that $\psi(n_1, \ldots, n_d) = 0$ whenever $n_d = 0$ or $n_d = 1$. From this we get $\psi = 0$. $\qquad\square$

4. APPLICATION TO UNIFORMLY QUASI-PERIODIC POTENTIALS

In this section we discuss potentials

$$(18) \qquad\qquad V(n_1, \ldots, n_d) = \lambda v(\beta_1 + n_1\omega_1, \ldots, \beta_d + n_d\omega_d)$$

with a continuous function v on $\mathbb{T}^d$, $\lambda > 0$, and $\beta, \omega \in \mathbb{T}^d$. We will put particular emphasis on the case $d = 2$, since this case was recently studied by Bourgain et al. in [3]. These authors were interested in proving localization, that is, pure point spectrum with exponentially decaying eigenfunctions, under suitable conditions on $v, \lambda, \beta, \omega$. Explicitly, they proved the following theorem.

Theorem 3 (Bourgain et al. [3])**.** *Let $d = 2$. Suppose that v is real-analytic and nonconstant on every horizontal and vertical line. Then, for every $\epsilon > 0$ and every $\beta \in \mathbb{T}^2$, there is a set $\mathcal{F}_\epsilon = \mathcal{F}_\epsilon(\beta) \subseteq \mathbb{T}^2$ with $\mathrm{mes}(\mathbb{T}^2 \setminus \mathcal{F}_\epsilon) < \epsilon$ such that for every $\omega \in \mathcal{F}_\epsilon$ and every $\lambda > \lambda_0(\epsilon, v)$ the operator $H = \Delta + V$ has pure point spectrum with exponentially decaying eigenfunctions.*

In the one-dimensional case, a similar theorem can be proved; see, for example, [2] or [14]. These one-dimensional results require certain diophantine conditions for ω. This is particularly explicit in Jitomirskaya [14], where she studies the almost Mathieu operator ($v = \cos$). This means that $\omega/(2\pi)$ should not be too well approximated by rational numbers. To complement this, Avron and Simon have shown that localization fails if ω is very well approximated by rational numbers. Indeed, call $\omega \in \mathbb{T}^1 \cong [0, 2\pi)$ a Liouville number if, for any $k \in \mathbb{N}$, there exists p_k, $q_k \in \mathbb{N}$ such that

$$\tag{19} \left| \frac{\omega}{2\pi} - \frac{p_k}{q_k} \right| \le k^{-q_k}.$$

Using Gordon's one-dimensional criterion from [12], Avron and Simon showed in [1] that for $v = \cos$ and any choice of λ, β, there are no eigenvalues. Their proof readily extends to more general v.

To complement Theorem 3 it would be nice to prove absence of eigenvalues if both ω_1 and ω_2 are Liouville numbers. We are not quite able to do this, but at least Theorem 2 gives the result if one of them is rational and the other one Liouville. This gives an uncountable set of exceptional $\omega \in \mathbb{T}^2$ where one cannot prove localization. More generally, we have the following theorem.

Theorem 4. *Suppose the potential V in (1) is of the form (18). Assume further that*

  (a) *$\omega_1, \ldots, \omega_{d-1}$ are rational and ω_d is Liouville, and*
  (b) *v is Lipschitz-continuous in the last component; that is, there is a constant L such that for every $x_1, \ldots, x_{d-1}, x_d, y_d \in T$ we have*

$$\tag{20} |v(x_1, \ldots, x_{d-1}, x_d) - v(x_1, \ldots, x_{d-1}, y_d)| \le L |x_d - y_d|.$$

Then, for every λ and every β, the operator H has no eigenvalues.

Proof. We verify the assumptions of Theorem 2. Assumption (i) of Theorem 2 follows from (a). To check (ii), we define the periodic approximants $V^{(m)}$ as follows. By choosing a suitable subsequence of p_k/q_k in (19) we may ensure that

$$\tag{21} \left| \frac{\omega_d}{2\pi} - \frac{p_{k_m}}{q_{k_m}} \right| \le \frac{1}{q_{k_m}} m^{-q_{k_m}}.$$

Let us set

$$V^{(m)}(n_1, \ldots, n_d) = \lambda v \left(\beta_1 + n_1 \omega_1, \ldots, \beta_{d-1} + n_{d-1}\omega_{d-1}, \beta_d + n_d (2\pi)\frac{p_{k_m}}{q_{k_m}} \right).$$

By (20) and (21) we have

$$\sup_{\substack{n = (n_1, \ldots, n_d) \in \mathbb{Z}^d: \\ (-2l+1)q_{k_m} \le n_d \le (2l)q_{k_m}}} |V(n) - V^{(m)}(n)| \quad \le \quad \lambda L (2l) q_{k_m} \left| \omega_d - 2\pi \frac{p_{k_m}}{q_{k_m}} \right|$$

$$\le \quad (4lL\pi) m^{-q_{k_m}}.$$

Thus (ii) of Theorem 2 holds, and we can conclude the proof. $\square$

In the case studied by Bourgain et al. we get the following corollary.

Corollary 4.1. *Let $d = 2$ and suppose that v is real-analytic. If one of ω_1, ω_2 is rational and the other one is Liouville, then for every λ and every θ, the operator H has no eigenvalues.*

Let us now explain why it is not possible to obtain the results of this section using the criterion from [13].

Proposition 4.2. *If V as in (18) is not periodic, then it cannot be approximated uniformly by periodic functions.*

Proof. Assume without loss of generality that V is not periodic in the direction $(0, \ldots, 0, 1)$. From this we can deduce that ω_d is irrational and that there exists some $(n_1, \ldots, n_{d-1})$ such that the map $n_d \mapsto V(n_1, \ldots, n_{d-1}, n_d)$ is nonconstant and hence aperiodic. Since v is continuous and orbits of irrational rotations on the torus $\mathbb{T}$ are dense, this map takes values that are dense in an interval $[a, b]$ with $\epsilon = b - a > 0$. Now assume that there is a periodic V_{per} with $\|V - V_{\text{per}}\|_{\ell^\infty(\mathbb{Z}^d)} < \epsilon/3$. Let p be its period in the direction $(0, \ldots, 0, 1)$. Then the map $n_d \mapsto V_{\text{per}}(n_1, \ldots, n_{d-1}, pn_d)$ is constant, whereas the map $n_d \mapsto V(n_1, \ldots, n_{d-1}, pn_d)$ again takes values which are dense in $[a, b]$ since also the rotation by $p\omega_d$ on $\mathbb{T}$ is irrational and has dense orbits. We therefore obtain a contradiction and conclude the proof. □

5. Application to Fibonacci-type superlattices

In this section we discuss Fibonacci-type superlattices, that is, models where the medium consists of finitely many types of (periodic) layers that are stacked according to a Fibonacci-type sequence: for example, a Sturmian sequence, a circle map, or a substitution sequence.

Let $V_0, \ldots, V_{N-1}$ be N periodic functions from $\mathbb{Z}^{d-1}$ to $\mathbb{R}$. These functions represent the N different types of layers. Now, for some sequence $x \in \{0, \ldots, N-1\}^{\mathbb{Z}}$, we define a potential $V : \mathbb{Z}^d \to \mathbb{R}$ by

$$(22) \qquad V(n_1, \ldots, n_{d-1}, n_d) = V_{x(n_d)}(n_1, \ldots, n_{d-1}).$$

Clearly, there are numbers $p_1, \ldots, p_{d-1} \in \mathbb{N}$ such that V is periodic with period p_k in direction $e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$, $1 \le k \le d-1$. Theorem 2 now gives the following:

Corollary 5.1. *Let $l = \prod_{k=1}^{d-1} p_k$. Suppose there are $n_k \in \mathbb{N}$ such that $n_k \to \infty$ and for every k we have, for every $-2l + 1 \le m \le 2l - 1$ and every $1 \le j \le n_k$,*

$$x(mn_k + j) = x(j).$$

Then the operator H with potential V as defined in (22) has empty point spectrum.

Let us now discuss concrete examples. Fibonacci sequences are given by $x(n) = \chi_{[1-\omega,1)}(\beta + n\omega \mod 1)$ with $\beta \in [0, 1)$ arbitrary and $\omega = (\sqrt{5} - 1)/2$ the inverse of the golden mean. Sturmian sequences are given by the same expression, but with a general irrational $\omega \in (0, 1)$. More generally, a circle map (also called coding of rotation) is given by $x(n) = \chi_{[1-\alpha,1)}(\beta + n\omega \mod 1)$ with $\alpha \in (0, 1)$ and $\beta \in [0, 1)$ arbitrary, and $\omega \in (0, 1)$ irrational. In all these examples we have $N = 2$. Examples with larger N will be discussed below.

Our first general theorem is the following.

Theorem 5. (a) *For every $\alpha \in (0,1)$ and almost every irrational $\omega \in (0,1)$, the set of $\theta \in [0,1)$ for which the circle map corresponding to the parameters α, ω, β satisfies the hypothesis of Corollary 5.1 has positive Lebesgue measure.*

More explicitly, if ω has continued fraction expansion coefficients a_n, $n \in \mathbb{N}$, obeying $\limsup_{n\to\infty} a_n \geq 4(4l-3)$, then x satisfies the assumption of Corollary 5.1 for every β from a set of positive Lebesgue measure.

(b) *For every $\alpha \in (0,1)$, almost every irrational $\omega \in (0,1)$, and almost every $\beta \in [0,1)$, the operator H with potential V given by (22) has empty point spectrum.*

Proof. (a) Fix α and some ω whose continued fraction expansion coefficients a_n, $n \in \mathbb{N}$, obey $\limsup_{n\to\infty} a_n \geq 4(4l-3)$. Write $x_\beta(n) = \chi_{[1-\alpha,1)}(\beta + n\omega \mod 1)$. Define the sets $G(n) \subseteq [0,1)$ by

$$G(n) = \{\beta \in [0,1) : x_\beta(mn+j) = x_\beta(j),\ -2l+1 \leq m \leq 2l-1,\ 1 \leq j \leq n\}.$$

Let $|\cdot|$ denote Lebesgue measure. Since $|\limsup G(n)| \geq \limsup |G(n)|$, it is sufficient to show that

$$(23) \qquad\qquad \limsup_{n\to\infty} |G(n)| > 0.$$

Recall that every irrational $\omega \in (0,1)$ has a continued fraction expansion

$$\omega = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

with uniquely determined $a_n \in \mathbb{N}$. The associated rational approximants $\frac{p_n}{q_n}$ are defined by

$$p_0 = 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2},$$
$$q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

We refer the reader to [16] for general information on continued fraction expansion theory. It is known that [16]

$$(24) \qquad\qquad \left| \omega - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \text{ for } n \geq 2$$

and [15]

$$(25) \qquad \limsup_{n\to\infty} \frac{q_{n+1}}{q_n} \geq \frac{a(\omega) + \sqrt{a(\omega)^2 + 4}}{2}\ , \text{ where } a(\omega) = \limsup_{n\to\infty} a_n.$$

We can now proceed similarly to Kaminaga [15]; see also [9] and [11]. Define

$$(26) \quad G_1(q_n) = \left\{ \beta : \min_{(-2l+2)q_n+1 \leq m \leq (2l-1)q_n} |\beta + m\omega - (1-\alpha)|_1 > |q_n\omega - p_n| \right\}$$

and

$$(27) \qquad G_2(q_n) = \left\{ \beta : \min_{(-2l+2)q_n+1 \leq m \leq (2l-1)q_n} |\beta + m\omega - 1|_1 > |q_n\omega - p_n| \right\},$$

where $|\cdot|_1$ denotes the distance from 0 on the torus $\mathbb{R}/\mathbb{Z}$. It follows from (24) that

$$(28) \qquad\qquad |(\beta + (m \pm q_n)\omega) - (\beta + m\omega)|_1 = |q_n\omega - p_n|.$$

From (26), (27), and (28) we get

(29) $$G(q_n) \subseteq G_1(q_n) \cap G_2(q_n),$$

and for $i = 1, 2$ we have

$$G_1(q_n)^c = \bigcup_{m=(-2l+2)q_n+1}^{(2l-1)q_n} \{\beta : |\beta + m\omega - (1-\alpha)|_1 \leq |q_n\omega - p_n|\}$$

and

$$G_1(q_n)^c = \bigcup_{m=(-2l+2)q_n+1}^{(2l-1)q_n} \{\beta : |\beta + m\omega - (1-\alpha)|_1 \leq |q_n\omega - p_n|\},$$

which, by (24), gives, for $i = 1, 2$,

(30) $$|G_i(q_n)^c| \leq (4l-3)q_n 2|q_n\omega - p_n| \leq 2(4l-3)\frac{q_n}{q_{n+1}}.$$

Combining (29) and (30), we get

(31) $$\limsup_{n\to\infty} |G(n)| \geq 1 - \frac{4(4l-3)}{\limsup_{n\to\infty} \frac{q_{n+1}}{q_n}}.$$

Thus we see from (25) and (31) that $\limsup a_n \geq 4(4l-3)$ is a sufficient condition for (23), and this condition is satisfied by almost every ω [16].

(b) By the general theory of ergodic families of Schrödinger operators (see, e.g., Carmona and Lacroix [5]), the point spectrum is nonrandom in the sense that it is the same set for μ-almost every element of the ergodic family, where μ is the ergodic measure associated with the family. The operators under consideration fit in this framework. Namely, relative to the shifts T_k, $1 \leq k \leq d$, in the d directions, the natural probability space is

$$\Omega = \{1, \ldots, p_1\} \times \cdot \times \{1, \ldots, p_{d-1}\} \times [0, 1)$$

and the natural probability measure is

$$\mu = \mu_{p_1} \times \cdots \times \mu_{p_{d-1}} \times \Lambda,$$

where μ_j is the uniform probability distribution on $\{1, \ldots, j\}$ and Λ is Lebesgue measure on $[0, 1)$. For an element $(m_1, \ldots, m_{d-1}, \beta) \in \Omega$, we define the operator $H^{(m_1,\ldots,m_{d-1},\beta)} = \Delta + V^{(m_1,\ldots,m_{d-1},\beta)}$, where the potential $V^{(m_1,\ldots,m_{d-1},\beta)}$ is given by

$$V^{(m_1,\ldots,m_{d-1},\beta)}(n_1, \ldots, n_{d-1}, n_d) = V_{x_\beta(n_d)}(n_1 + m_1, \ldots, n_{d-1} + m_{d-1}).$$

Part (a) and Corollary 5.1 show that under the condition $\limsup a_n \geq 4(4l-3)$ the set of $(m_1, \ldots, m_{d-1}, \beta) \in \Omega$ for which $H^{(m_1,\ldots,m_{d-1},\beta)}$ has empty point spectrum has positive μ-measure. Since there is a set Σ_{pp} such that $\sigma_{pp}(H^{(m_1,\ldots,m_{d-1},\beta)}) = \Sigma_{pp}$ for μ-almost every $(m_1, \ldots, m_{d-1}, \beta) \in \Omega$ (see [5]), the assertion follows. $\square$

Notice that the above theorem does not apply to the Fibonacci case, since in this case we have $a_n = 1$ for every $n \in \mathbb{N}$. The following theorem, which proves a similar result for all Sturmian cases (and hence applies also to the Fibonacci case) provided that the layer potentials are constant, is therefore of some interest.

Theorem 6. *Assume that the layer potentials V_0, V_1 are constant. For every irrational $\omega \in (0,1)$, and almost every $\beta \in [0,1)$, the operator H with potential V given by (22), where x is the Sturmian sequence corresponding to parameters ω, β, has empty point spectrum.*

Proof. We have $l = 1$ and hence a situation similar to the one-dimensional case. Thus, the theorem can be proved in the same way as the corresponding result in the one-dimensional case; see Kaminaga [15] for details.   □

Of course, we can also consider a more general model with N different types of layers, where we have a coding of a rotation using a partition of the torus into N, rather than two, intervals. With a similar method one can prove a result analogous to Theorem 5. For more details, we refer the reader to the paper of Delyon and Petritis [11] (treating the one-dimensional case).

Another class of examples is given by models generated by sequences x that belong to a substitution subshift. Pick some $N \in \mathbb{N}$ and denote $A = \{0, \ldots, N-1\}$. A substitution S is a mapping from A to the set A^* of finite words over A. By concatenation it can naturally be extended to A^* and $A^{\mathbb{N}}$. A fixed point $u \in A^{\mathbb{N}}$ is called a substitution sequence associated with S. The substitution is called primitive if there is a power S^k of S such that for every pair $a, b \in A$, $S^k(a)$ contains b. Suppose we are given a substitution sequence u associated with a primitive substitution S. We define the subshift X to be the set of all two-sided sequences $x \in A^{\mathbb{Z}}$ such that every finite subword of x is a subword of u. It is known that (X, T) is a strictly ergodic subshift, where T is the standard shift on $A^{\mathbb{Z}}$ [21]. Let us call the unique ergodic measure ν. For example, if $N = 2$, $S(0) = 1$, $S(1) = 10$, then $u = 1011010110110\ldots$ is the unique fixed point of S in $A^{\mathbb{N}}$ and X (essentially) consists of the Fibonacci sequences $x(n) = \chi_{[1-\omega,1)}(\beta + n\omega \mod 1)$ (X contains one additional orbit, but this orbit has of course zero ν-measure). This is why substitution-generated models are also a natural generalization of models generated by Fibonacci sequences.

Now let $V_0, \ldots, V_{N-1}$ be given and consider for $x \in S$ the potential (22). Define l as above. Then we have the following theorem.

Theorem 7. *Let S be a primitive substitution, let u be a fixed point of S, and denote by X the associated subshift. Suppose that u contains a $4l$-th power, that is, there is a nonempty word w such that w^{4l} is a subword of u. Then for ν-almost every sequence $x \in X$, the operator H has empty point spectrum.*

Proof. We show that for x from a set of positive ν-measure, the assumption of Corollary 5.1 is satisfied. The assertion of the theorem then follows as in the proof of Theorem 5 (b).

To do so, we can use a modified version of the argument given in [7] for the one-dimensional case. We will be sketchy at some points and refer the reader to [7] for more details. Let us first recall some properties of primitive substitutions; compare [21]. The substitution matrix $M(S) = (m_{ij})_{1 \leq i,j \leq s}$ is defined by $m_{ij} = \#_{a_i} S(a_j)$, where for $v_1, v_2 \in A^*$, $\#_{v_1} v_2$ denotes the number of occurrences of v_1 in v_2. The Perron-Frobenius theorem yields the existence of a dominant eigenvalue θ which can be regarded as an asymptotic blow-up factor: For every $v \in A^*$, there exists $c(v) > 0$ such that $\lim_{n \to \infty} \frac{|S^n(v)|}{\theta^n} = c(v)$.

Another key property of substitution sequences arising from primitive substitution is the existence of positive frequencies of words: For any $v \in A^*$ which occurs in a fixed point of S and any $b \in A$, the limits

$$d(v) = \lim_{n \to \infty} \frac{\#_v S^n(b)}{|S^n(b)|} > 0$$

exist and are independent of b. Finally, for cylinder sets $[b_0 \ldots b_{l-1}]_{[m,m+l-1]} = \{x \in X : x(m+i) = b_i, \ 0 \le i \le l-1\}$, the ν-measure of such a set is simply given by the frequency of the word $b_0 \ldots b_{l-1}$, $\nu([b_0 \ldots b_{l-1}]_{[m,m+l-1]}) = d(b_0 \ldots b_{l-1})$.

Now define

$$G(n) = \{x \in X \ : \ x(k) = x(mn+j), \ -2l+1 \le m \le 2l-1, \ 1 \le j \le n\}.$$

We want to show that $\limsup \nu(G(n)) > 0$, which implies $\nu(\limsup G(n)) > 0$. Define $n_k = |S^k(w)|$ and, for $M \in \mathbb{N}$, $P_M : A^M \to A^M$, $b_1 b_2 \ldots b_M \mapsto b_2 \ldots b_M b_1$. We will obtain estimates for the ν-measure of $G(n)$ on the subsequence $(n_k)_{k \in \mathbb{N}}$. This is natural since, by self-similarity, the occurrence of w^{4l} in u yields existence of $(4l)$-block structures of length $4n_k$ in u. Thus, we have plenty of $(4l-1)$-block structures of length $(4l-1)n_k$ if we consider cyclic permutations of $S^k(w)$. Define $r_k = \min\{i \ge 1 : S^k(v) = P_{n_k}^i(S^k(v))\}$. Obviously, $1 \le r_k \le n_k$. Clearly, there exists $d_k \in \mathbb{N}$ such that $n_k = d_k \cdot r_k$. It is relatively easy to see that for $0 \le i < r_k$, we have

$$d((P_{n_k}^i(S^k(w)))^{4l-1}) \ge \frac{d_k}{\theta^k} \cdot d(w^{4l}).$$

Putting everything together, this yields

$$\limsup \nu(G(n_k)) \ge c(v) \cdot d(w^{4l}) > 0,$$

concluding the proof. $\qquad\square$

Finally we explain why the results of this section cannot also be established using the criterion of [13].

Proposition 5.2. *If V as in (22) is not periodic, then it cannot be approximated uniformly by periodic functions.*

Proof. Since V takes only finitely many values, there is a minimum distance separating each pair of them. Thus, once we have sufficiently good uniform approximation of V by a periodic function, the approximand must coincide with V and hence V is periodic. $\qquad\square$

6. Concluding remarks

Theorem 2 can also be proved by means of a direct integral decomposition (cf., e.g., [10] for this approach in a different context). In fact, the proof given in Section 3 is related to this approach but slightly more direct. The direct integral decomposition could in principle give more detailed results about the spectral type of H, provided that one is able to study the fiber operators in sufficient detail. These fiber operators are, as we have essentially seen in Section 3, operators on the strip. Unfortunately, quasi-periodic operators on the strip are presently not sufficiently well understood, except for those where localization occurs; see, for example, Bourgain and Jitomirskaya [4] and Klein et al. [17]. For those operators with empty point spectrum which we have discussed in this article, one is

currently not able to distinguish between singular continuous spectrum and absolutely continuous spectrum in the same generality as for strictly one-dimensional models. This is due to the lack of a sufficiently strong analog of Kotani theory [18], [19], which is among the central tools for studying the absolutely continuous spectrum of (ergodic; e.g., quasi-periodic) one-dimensional Schrödinger operators. Kotani and Simon have dealt with this issue in [20], but the results for the strip of width greater than one are much weaker than those in the case of width one. Explicitly, it is known that the one-dimensional analogs of the operators studied in Sections 4 and 5 have singular continuous spectrum (cf. [8] for a survey of results for one-dimensional Fibonacci-type models), whereas this is not known for strips of nontrivial width. Absence of absolutely continuous spectrum was established in both cases using Kotani theory [1], [19].

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